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The locomotion of elongated bodies in pipes

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The theory describing the swimming mechanism of an elongated body is extended to cover the cases of locomotion through unsteady streams in pipes. Such an extension is essential for the artificial fish 'Pod', a medical device which swims in the patient's blood vessel. Two approaches are considered. First, potential theory is considered, and the results achieved show that the main influence of the pipe is on evaluation of the proper virtual mass. Next the flow is assumed to be viscous. The consideration of viscosity is obviously necessary for flows in pipes. In that case the virtual mass is replaced by another equivalent mass depending on the viscosity and on the angular frequency of the lateral motion and in addition new terms appear in the local lift expressions. These are recognized as the viscous damping force.

1. Introduction

Many articles trying to explain the mechanism of the swimming of fish and the locomotion of different sea animals have been published during the past ten to twenty years. Special emphasis has been given to elongated bodies performing undulatory motion in an attempt to propel themselves through liquid media. For such bodies slender-body theory was applied in different ways to set up mathematical models and to analyse the propulsion capabilities of the bodies. The pioneer work in this field is undoubtedly related to Lighthill (1960), where a potential flow field was assumed for high Reynolds number cases. Since then the basic theory has been enlarged to cover many other cases such as the contribution of fins, combinations of two-dimensional tails with elongated bodies and large amplitude theory of fish locomotion (see Lighthill 1971).

Strong viscous effects were considered by Lavie (see Lavie 1970, 1972) for the locomotion of elongated bodies in an infinite liquid. Slender-body theory, when applied to that case, led to relatively simplified two-dimensional cross-flow equations with parameters depending on the longitudinal distance. The Oseen approximation was used for the cross-flow field. The validity of this approximation and the boundary conditions assumed are discussed in these articles.

The problem of the swimming of an elongated body in pipes or tubes is of essential importance for artificial fish such as the Pod (see Lavie 1966), a medical device which swims in a patient's blood vessel. This article deals with such a problem. For convenience we choose the pipe's cross-section to be circular although the same theory can be easily extended to other cross-sections. To do that, as can be seen later on, one has to find the potential and the stream functions for the two-dimensional cylinder whose instantaneous cross-section oscillates between the walls of the pipe. In addition to that we assume here that the mainstream in the pipe is of constant velocity and an oscillating longitudinal velocity component which represents the variation of the blood flow in the vessels.

In this article we approach the problem in two ways. At the beginning we assume the flow to be purely potential. Then Lighthill's (1970) model is extended and it is shown that when the radius of the pipe becomes infinite the two cases are identical. In the following part of the article we assume the flow to be viscous and the work of Lavie (1970, 1972) is extended. It is shown then that the hydrodynamic virtual mass has to be replaced by other equivalent masses depending also on the viscosity and the angular frequency of the lateral motion. The expressions for these masses consist of Bessel functions of the first and second kind. Lastly it is shown that when the viscosity is equal to zero the two approaches coincide. The relevance of potential-flow theory to the Pod is somehow doubtful. The Reynolds number based on the Pod's diameter and its lateral velocity may vary between 10 and 10^3. However, the potential flow has some interesting features and at the end of the article it is shown how as the Reynolds number goes to infinity the potential and viscous theories coincide.

2. Potential theory

2.1. *The potential flow*

Figure 1 shows an elongated flexible body of length l propelling itself in a liquid streaming in a pipe of radius b. The body swims into the $-x$ direction with velocity $U(t)$, which in general changes with time. The origin of the co-ordinates is fixed at the nose of the body and the stream far from it (denoted by point 1) is potential with uniform velocity $W(t)$ distributed along the pipe's cross-section. The transverse motion of the body's centre-line in the z direction is $h(x,t)$, which depends on the distance x from the origin and on the time t. The instantaneous cross-sectional area of the body is $A(x)$.

We shall assume that in the general case the velocities $W(t)$ and $U(t)$ have the form

$$W(t) = W_0 + W' e^{i\omega t}, \quad U(t) = U_0 + U'(t), \tag{1}$$

where W_0 and U_0 are average velocities, $W' e^{i\omega t}$ is some periodic oscillatory velocity along the longitudinal axis and $U'(t)$ is the time-dependent part of the velocity of the body, which is included to satisfy the equations of motion.

To make the body fixed in the space we superimpose on the fluid-body system a velocity $U(t)$. In order to consider the new conditions at point 1 we assume that $A(x)$ changes very slowly along the body so that the new velocity at point 1 is approximately

$$W_1(t) = W(t) + U(t)\frac{A - \bar{A}}{A}, \quad \bar{A} = \frac{1}{l}\int_0^l A(x)\,dx, \tag{2}$$

where A is the inner pipe cross-sectional area and $\bar{A}$ is the average body cross-sectional area. When $A \to \infty$ the fluid space becomes infinite and

$$W_1(t) = W(t) + U(t).$$

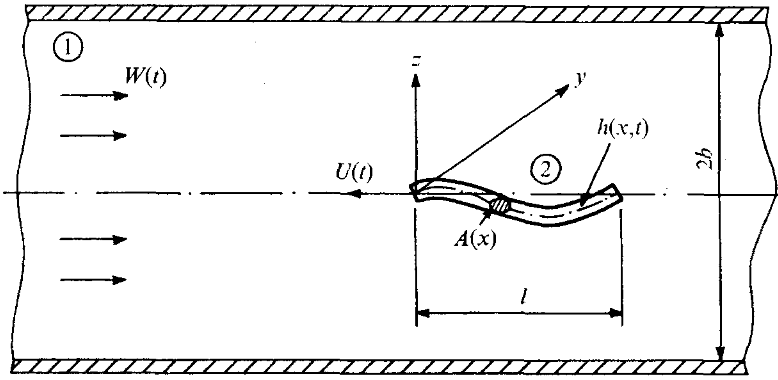

FIGURE 1. Elongated body swimming in a pipe in a fluid with variable velocity.

If Φ_1 and p_1 are the potential and the pressure at point 1, then

$$\left.\begin{aligned} \Phi_1 &= xW_1(t), \\ \frac{1}{\rho}\frac{\partial p_1}{\partial x} &= -\frac{\partial W_1}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\partial \Phi_1}{\partial x}\right), \\ p_1 &= -\rho(\partial\Phi_1/\partial t)+p_0(t) \end{aligned}\right\} \tag{3}$$

and from Bernoulli's equation one gets

$$H = \frac{p_1}{\rho}+\frac{\partial \Phi_1}{\partial t}+\tfrac{1}{2}W_1(t)^2 = \text{constant},$$

where H is the total weight and $p_0(t)$ is a pressure which depends on time. Thus

$$\rho H = p_0+\tfrac{1}{2}\rho W_1(t)^2 = p_2+\rho(\partial\Phi_2/\partial t)+\tfrac{1}{2}\rho W_2(t)^2 = \text{constant},$$

where p_2, Φ_2 and W_2 are the fluid properties at point 2, not far from the body (see figure 1).

Following Lighthill (1960) we introduce new co-ordinates X, Y, Z and T, where

$$X = x, \quad Y = y, \quad Z = z-h(x,t), \quad T = t, \tag{4}$$

so that $$\frac{\partial}{\partial x} = \frac{\partial}{\partial X}-\frac{\partial h}{\partial x}\frac{\partial}{\partial Z}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial Z}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial T}-\frac{\partial h}{\partial t}\frac{\partial}{\partial Z}.$$

At point 2 the longitudinal velocity of the fluid and its potential are

$$\left.\begin{aligned} W_2(X,T) &= U(t)+W(T)\,A/(A-A(x)), \\ \Phi_2 &= \phi_0(X,T)+\phi_1(X,Y,Z,T)+\phi_2(X,Y,Z,T), \\ \partial\phi_0/\partial X &= W_2(X,T). \end{aligned}\right\} \tag{5}$$

$\phi_0+\phi_1$ are the potentials when $h(X,T) = 0$ and ϕ_2 is the potential due to the transverse motion $h(x,t)$. ϕ_0 is the potential of the free stream. Following Lighthill (1960) and using the slenderness parameter ϵ we find that the transformed Laplace equation for the potential Φ_2 reduces to

$$\frac{\partial^2\Phi_2}{\partial Y^2}+\frac{\partial^2\Phi_2}{\partial Z^2} = -\frac{\partial W_2}{\partial X} \tag{6}$$

when terms of order ϵ^2 are ignored. Notice that for an infinite fluid $\partial W_2/\partial X = 0$. Since $\partial^2\phi_0/\partial X^2 = \partial W_2/\partial x$ it follows that (6) can be split:

$$\frac{\partial^2\phi_1}{\partial Y^2}+\frac{\partial^2\phi_1}{\partial Z^2} = -\frac{\partial W_2}{\partial X}, \quad \frac{\partial^2\phi_2}{\partial Y^2}+\frac{\partial^2\phi_2}{\partial Z^2} = 0. \tag{7}$$

If the equation of the surface of the stretched-straight body is $F(x,y,z) = 0$ then in the co-ordinates (4) the surface of the swimming body has the equation $F(X, Y, Z) = 0$, and the boundary condition on it is obtained by setting the substantial derivative DF/DT equal to zero:

$$\frac{DF}{DT} = -\frac{\partial h}{\partial T}\frac{\partial F}{\partial Z}+\left(\frac{\partial \Phi_2}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial \Phi_2}{\partial Z}\right)\left(\frac{\partial F}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial F}{\partial Z}\right)+\frac{\partial \Phi_2}{\partial Y}\frac{\partial F}{\partial Y}+\frac{\partial \Phi_2}{\partial Z}\frac{\partial F}{\partial Z} = 0.$$

Under the assumption that $\partial h/\partial T$ and $\partial h/\partial X$ are small the above expression can be simplified to

$$-\frac{\partial h}{\partial T}\frac{\partial F}{\partial Z}+W_2\left(\frac{\partial F}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial F}{\partial Z}\right)+\frac{\partial \Phi_2}{\partial Y}\frac{\partial F}{\partial Y}+\frac{\partial \Phi_2}{\partial Z}\frac{\partial F}{\partial Z} = 0. \tag{8}$$

One boundary condition is obtained when $h(X,T) = 0$ and $\phi_2 = 0$, then

$$-W_2\frac{\partial F}{\partial X} = \frac{\partial \phi_1}{\partial Y}\frac{\partial F}{\partial Y}+\frac{\partial \phi_1}{\partial Z}\frac{\partial F}{\partial Z}. \tag{9}$$

The second boundary condition is achieved when $h(X, T) \neq 0$ and $\phi_2 \neq 0$ by subtracting (9) from (8):

$$\frac{\partial F}{\partial Z}\left(\frac{\partial \phi_2}{\partial Z}-\frac{\partial h}{\partial T}-W_2\frac{\partial h}{\partial X}\right)+\frac{\partial \phi_2}{\partial Y}\frac{\partial F}{\partial Y} = 0. \tag{10}$$

The third boundary condition is that on the surface of the pipe:

$$\partial\Phi_2/\partial r = 0 \quad \text{on} \quad r = b, \tag{11}$$

when r is the radius in cylindrical co-ordinates.

2.2. *The lift distribution*

In the new co-ordinates ϕ_2 is the potential at given X of an infinite cylinder with cross-sectional area $A(x)$ moving in the pipe along the Z axis with transverse velocity

$$V(X,T) = (\partial h/\partial T)+W_2(\partial h/\partial X). \tag{12}$$

Hence, if $\phi(X, Y, Z)$ is the potential of an infinite cylinder with cross-sectional area $A(X)$ moving with unit velocity in the Z direction then

$$\phi_2(X, Y, Z, T) = V(X,T)\,\phi(X, Y, Z). \tag{13}$$

The pressure at point 2 can be calculated from Bernoulli's equation. When terms of order ϵ^4 are neglected with respect to terms of order ϵ^2 one gets

$$p_2-p_0 = p_{\mathrm{I}}+p_{\mathrm{II}}+p_{\mathrm{III}}, \tag{14a}$$

$$p_{\mathrm{I}} = \rho\left\{-\frac{\partial \phi_0}{\partial T}-\frac{\partial \phi_1}{\partial T}+\tfrac{1}{2}\rho(W_1^2-W_2^2)-W_2\frac{\partial \phi_1}{\partial X}-\frac{1}{2}\left(\frac{\partial \phi_1}{\partial Y}\right)^2-\frac{1}{2}\left(\frac{\partial \phi_1}{\partial Z}\right)^2\right\}, \tag{14b}$$

$$p_{\mathrm{II}} = \rho\left\{-\frac{\partial \phi_2}{\partial T}-W_2\frac{\partial \phi_2}{\partial X}+\left(V-\frac{\partial \phi_2}{\partial Z}\right)\frac{\partial \phi_1}{\partial Z}-\frac{\partial \phi_1}{\partial Y}\frac{\partial \phi_2}{\partial Y}\right\}, \tag{14c}$$

$$p_{\mathrm{III}} = \rho\left\{V\frac{\partial \phi_2}{\partial Z}-\frac{1}{2}\left(\frac{\partial \phi_2}{\partial Y}\right)^2-\frac{1}{2}\left(\frac{\partial \phi_2}{\partial Z}\right)^2\right\}. \tag{14d}$$

Lighthill (1960) showed that p_{I} and p_{III} do not produce local lift for symmetric cross-sections. The only lift is produced by p_{II} and then

$$L(X,T) = -\rho\oint_{s_x}\left\{\frac{\partial\phi_2}{\partial T}+W_2\frac{\partial\phi_2}{\partial X}+\left(\frac{\partial\phi_2}{\partial Z}-V\right)\frac{\partial\phi_1}{\partial Z}+\frac{\partial\phi_1}{\partial Y}\frac{\partial\phi_2}{\partial Y}\right\}dY, \tag{15}$$

where L is the local lift at given X and s_x is the contour of $A(X)$. To solve (15) we remember that $\phi_2 = V\phi$ and in addition

$$\rho\oint_{s_x}\phi(X,Y,Z)\,dY = \rho\tilde{A}(X), \tag{16}$$

where $\rho\tilde{A}(X)$ is the virtual mass per unit length of the body swimming in the pipe. If for instance the local cross-section is circular with radius $a(x)$ then

$$\partial\phi_2/\partial r = \begin{cases} -V\cos\theta & \text{at} \quad r = a(x), \\ 0 & \text{at} \quad r = b. \end{cases}$$

θ is the polar angle measured from the Z axis. Hence

$$\phi_2 = \frac{a(x)^2}{b^2-a(x)^2}V\cos\theta\left(r+\frac{b^2}{r}\right).$$

The hydrodynamical forces F_y and F_Z are calculated from

$$p/\rho = (\partial\phi_2/\partial t)-\tfrac{1}{2}q^2 \quad (q \text{ is the total velocity}),$$

$$F_Z = -\oint_{s_x} lp\,ds = -\rho\frac{b^2+a(x)^2}{b^2-a(x)^2}\pi a(x)^2\frac{\partial V}{\partial t},$$

$$F_y = -\oint_{s_x} mp\,ds = 0,$$

where l and m are the direction cosines and ds is the element of arc along the contour s_x. In that case one sees immediately that the virtual mass per unit length is

$$\rho\tilde{A}(x) = \rho\frac{b^2+a(x)^2}{b^2-a(x)^2}A(x), \quad A(x) = \pi a(x)^2. \tag{17}$$

The evaluation of the local lift L from (15) is very similar to that provided in the appendix of Lighthill (1960). A more exact method is given in Lavie (1973). The result here is certainly different from that for an infinite fluid and is found to be

$$L(X,T) = -\rho\left\{\frac{\partial}{\partial T}+W_2\frac{\partial}{\partial X}\right\}(V\tilde{A}(X))-\rho V\frac{\partial W_2}{\partial X}\tilde{A}(X). \tag{18}$$

For an infinite fluid ($b\to\infty$) $W_2 \to U(t)+W(t) = W_1$, $\partial W_2/\partial X = 0$ and the results agree definitely with Lighthill (1960).

2.3. *The thrust force*

The thrust P is written as an integral over the surface of the swimming body and then expressed using the co-ordinates (4) as an integral over the surface s:

$$P = \iint(p_{\mathrm{I}}+p_{\mathrm{II}}+p_{\mathrm{III}})\,dy\,dz = \int_s\int(p_{\mathrm{I}}+p_{\mathrm{II}}+p_{\mathrm{III}}))\,dY\left(\frac{\partial h}{\partial X}dX+dZ\right)$$
$$= \int_0^l L(X,T)\frac{\partial h}{\partial X}+\int_s\int(p_{\mathrm{I}}+p_{\mathrm{II}}+p_{\mathrm{III}})\,dY\,dZ. \tag{19}$$

For a cylinder of symmetric cross-section the pressure p_{II} does not provide any resultant force in the x direction. The pressure p_{I} associated with the flow when $h(x,t) = 0$ creates a force which depends on the variation of the potential with time. Thus

$$\int_s\int p_{\mathrm{I}}\,dY\,dZ = -\rho\int_s\int \frac{\partial}{\partial T}(\phi_0+\phi_1)\,dY\,dZ.$$

The value of the above integral depends on the body's shape and on the radius of the pipe. For an ellipsoid $(X^2/a^2)+(Y^2/b^2)+(Z^2/c^2) = 1$, for instance, and for an infinite fluid ($b\to\infty$) the fluid velocity becomes $W_2(T) = U(T)+W(T)$ and the thrust force resulting from the above integral is

$$\iint p_{\mathrm{I}}\,dY\,dZ = kM'\frac{\partial W_2}{\partial T},$$

where M' is the displaced mass and

$$k_2 = \frac{\alpha_0}{2-\alpha_0}, \quad \alpha_0 = abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)\,\Delta},$$
$$\Delta = \{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{\frac{1}{2}}.$$

When the radius of the pipe is finite then $W_2(X,T) = U(T)+W(T)\,[A/(A-A(X)]$ will be influenced by the body's cross-section as well. In general we can write

$$\int_s\int p_{\mathrm{I}}\,dY\,dZ = -\rho\int_s\int\frac{\partial}{\partial T}(\phi_0+\phi_1)\,dY\,dZ = M'\left\{k_1\frac{\partial U}{\partial T}+k_2\frac{\partial W}{\partial T}\right\}, \tag{20}$$

where k_1 is a hydrodynamic coefficient influenced by the body shape and k_2 also depends on the radius of the pipe.

The integral of the local lift provides

$$\begin{aligned}\int_0^l L(X,T)\frac{\partial h}{\partial X}\,dX &= -\rho\int_0^l \frac{\partial h}{\partial X}\left\{\frac{\partial}{\partial T}[V\tilde{A}(X)]+\frac{\partial}{\partial X}[W_2 V\tilde{A}(X)]\right\}dX\\ &= -\rho\frac{\partial}{\partial T}\int_0^l\frac{\partial h}{\partial X}V\tilde{A}(X)\,dX-\rho\int_0^l\frac{\partial W_2}{\partial X}\frac{\partial h}{\partial X}V\tilde{A}(X)\,dX\\ &\quad -\rho\left[\frac{\partial h}{\partial X}W_2V\tilde{A}(X)\right]_0^l+\tfrac{1}{2}\rho[V^2\tilde{A}(X)]_0^l-\tfrac{1}{2}\rho\int_0^l V^2\frac{\partial\tilde{A}}{\partial X}\,dX. \end{aligned}\tag{21}$$

(More details are given in Lavie 1973.)

To evaluate $\iint p_{\mathrm{III}}\,dY\,dZ$ we evaluate the change of the fluid momentum between the cross-section at $\tilde{A}(X+\delta X)$ and at $\tilde{A}(X)$. Following Lavie (1973) we find that the change of the momentum is

$$\begin{aligned}\rho V[A(X+\delta X)-\tilde{A}(X)]&+\rho\tilde{A}(X)[V(X+\delta X)-V(X)]\\ &= \rho V\frac{\tilde{A}(X)}{dX}\,dX+\rho\tilde{A}(X)\frac{\partial W_2}{\partial X}\frac{\partial h}{\partial X}\,dX\end{aligned}$$

and as explained by Lighthill (1960) we finally have

$$\int_s\int p_{\mathrm{III}}\,dY\,dZ = \int_0^l\left[\tfrac{1}{2}\rho V^2\frac{\partial\tilde{A}(X)}{\partial X}\,dX+\rho V\tilde{A}(X)\frac{\partial W_2}{\partial X}\frac{\partial h}{\partial X}\,dX\right]. \tag{22}$$

The overall thrust force is therefore

$$P = M'\left\{k_1\frac{\partial U}{\partial T}+k_2\frac{\partial W}{\partial T}\right\}-\rho\frac{\partial}{\partial T}\int_0^l\frac{\partial h}{\partial X}V\tilde{A}(X)\,dX$$
$$-\rho\left[\frac{\partial h}{\partial X}W_2V\tilde{A}(X)\right]_0^l+\tfrac{1}{2}\rho[V^2\tilde{A}(X)]_0^l. \quad (23)$$

To find the mean thrust by averaging over a long time period (indicated by an overbar) we remember that

$$U = U_0+U'(T), \quad W_2 = U_0+U'(T)+[W_0+W'e^{i\omega t}]A/(A-A(X)),$$

thus
$$\overline{P} = \tfrac{1}{2}\rho\left\{\tilde{A}(X)\overline{\left(\frac{\partial h}{\partial T}\right)^2}-\left[U_0^2+\overline{U'^2}+(W_0^2+\tfrac{1}{2}\overline{W'^2})\left(\frac{A}{A-A(X)}\right)^2\right.\right.$$
$$\left.\left.+2U_0W_0\frac{A}{A-A(X)}\right]\tilde{A}(X)\overline{\left(\frac{\partial h}{\partial X}\right)^2}\right\}_0^l. \quad (24)$$

3. Viscosity considerations

3.1. *Formulation of the problem*

In many cases the viscosity has to be considered very carefully, especially in cases where the diameter of the pipe is not much bigger than the cross-sectional dimensions of the swimming body. The velocity of the mean stream is distributed between the inner wall of the pipe and the surface area of the body. We assume that at the edge of the body's boundary layer the mean velocity is again

$$W_2 = U+WA/(A-A(X))$$

although more accurate calculations have to be done in any particular case. Following Lavie (1970) we shall assume that the boundary conditions may be expressed as

$$v = \begin{cases}0,\\ 0,\end{cases} \quad w = \begin{cases}V = (\partial h/\partial t)+W_2(\partial h/\partial x) & \text{on the cylinder,}\\ 0 \quad \text{on the pipe.}\end{cases} \quad (25)$$

Since the body is elongated we can use the assumptions of slender-body theory and write the equation of continuity in the following way:

$$\frac{du}{dx}+\frac{dv}{dz}+\frac{dw}{dv} \sim \frac{dW_2}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = 0. \quad (26)$$

In what follows we shall use the linearized Oseen equations, where the velocity u is taken as W_2+u' and u' is the perturbation velocity. This explains the use of dW_2/dx in (26). The form of (26) suggests that we can split it in the following way:

$$v = v_0+v_1, \quad w = w_0+w_1, \quad p = p_0+p_1,$$
$$\frac{dv_0}{dy}+\frac{dw_0}{dz} = -\frac{\partial W_2}{dx}, \quad \frac{dv_1}{dy}+\frac{dw_1}{dz} = 0. \quad (27)$$

It follows that v_1 and w_1 are the velocities of a two-dimensional flow characterized by the stream function ψ such that

$$v_1 = \partial\psi/\partial z, \quad w_1 = -\partial\psi/\partial y. \quad (28)$$

Lavie (1972) has explained the validity of the Oseen equations when the influence of the viscosity on the lift and on the thrust force is considered. Using the Oseen equations we can write the following sets of differential equations:

$$\frac{\partial W_2}{\partial t}+W_2\frac{\partial W_2}{\partial x}+\frac{1}{\rho}\frac{\partial p_0}{\partial x}=0, \tag{29a}$$

$$\frac{\partial v_{01}}{\partial t}+W_2\frac{\partial v_{01}}{\partial x}+\frac{1}{\rho}\frac{\partial p_0}{\partial y}=0, \tag{29b}$$

$$\frac{\partial w_{01}}{\partial t}+W_2\frac{\partial w_{01}}{\partial x}+\frac{1}{\rho}\frac{\partial p_0}{\partial z}=0, \tag{29c}$$

$$\Delta p_0=0; \tag{29d}$$

$$\frac{\partial u'}{\partial t}+W_2\frac{\partial u'}{\partial x}-\nu\Delta u'=0, \tag{30a}$$

$$\frac{\partial v_{02}}{\partial t}+W_2\frac{\partial v_{02}}{\partial x}-\nu\Delta v_{02}=0, \tag{30b}$$

$$\frac{\partial w_{02}}{\partial t}+W_2\frac{\partial w_{02}}{\partial x}-\nu\Delta w_{02}=0; \tag{30c}$$

$$\frac{\partial v_1}{\partial t}+W_2\frac{\partial v_1}{\partial x}+\frac{1}{\rho}\frac{\partial p_1}{\partial y}-\nu\Delta_2 v_1=0, \tag{31a}$$

$$\frac{\partial w_1}{\partial t}+W_2\frac{\partial w_1}{\partial x}+\frac{1}{\rho}\frac{\partial p_1}{\partial z}-\nu\Delta_2 u_1=0, \tag{31b}$$

$$\Delta_2 p_1=0; \tag{31c}$$

where Δ is the three-dimensional Laplacian operator $(\partial^2/\partial x^2)+(\partial^2/\partial y^2)+(\partial^2/\partial z^2)$ and Δ_2 is the two-dimensional Laplacian operator $(\partial^2/\partial y^2)+(\partial^2/\partial z^2)$.

In the above equations it is understood that $u=W_2+u'$, $v=v_{01}+v_{02}+v_1$ and $w=w_{01}+w_{02}+w_1$. Equations (30) determine the skin friction on the body. Since the skin friction for this case is not treated here we shall leave those equations.

3.2. *The lift and the thrust*

The pressure p_0 in (28) is similar to the pressure p_{I} in the potential flow (see equation (19)). This pressure does not provide local lift because we assume the cross-section to be symmetric. It does provide a thrust force. If Φ is the potential function such that $\partial\Phi/\partial x=-W_2$, $\partial\Phi/\partial y=-v_{01}$ and $\partial\Phi/\partial z=-w_{01}$ then

$$-\frac{1}{\rho}p_0=-\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}(W_2^2+v_{01}^2+w_{01}^2)\sim-\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}W_2^2. \tag{32}$$

This pressure gives the same thrust force P_0 as in (20), i.e.

$$P_0=\int_S\int p_0\,dy\,dz=M'\left(k_1\frac{\partial U}{\partial t}+k_2\frac{\partial W}{\partial t}\right). \tag{33}$$

Finally we look for a solution of (31) in order to calculate the local lift and the total thrust.

Following Lavie (1970) and using (28) we define

$$\psi = V\psi', \quad V = (\partial h/\partial t) + W_2(\partial h/\partial x) \tag{34}$$

and we again get the linearized Oseen equation

$$\Delta_2\left(V\Delta_2\psi' - \left(\frac{\partial}{\partial t} + W_2\frac{\partial}{\partial x}\right)V\psi'\right) = 0.$$

For transverse motion $h(x,t)$ of the simple harmonic type it can be shown that

$$\frac{(\partial V/\partial t) + W_2(\partial V/\partial x)}{V} \sim i(\Omega - kW_2) = \nu c^2, \tag{35}$$

where Ω is the angular velocity and k is the wavenumber of the travelling wave. Thus we can solve (31) by writing

$$\left.\begin{aligned}\Delta_2(\Delta_2\psi' - c^2\psi') = 0, \quad \psi' = \psi_1 + \psi_2,\\ \Delta_2\psi_1 = 0, \quad \Delta_2\psi_2 - c^2\psi_2 = 0.\end{aligned}\right\} \tag{36}$$

The solution for ψ_1 suggests that there exists a potential $\Phi_1(x,y,z,t)$ conjugate to ψ_1 such that

$$\left.\begin{aligned}&\Phi_1 = V\phi, \quad \Delta_2\Phi_1 = 0,\\ &p_1 = \rho[(\partial/\partial t) + W_2(\partial/\partial x)](V\phi),\\ &\Gamma_1 = \tfrac{1}{2}\Delta_2\psi = \tfrac{1}{2}V\Delta_2\psi_2 = -\tfrac{1}{2}c^2V\psi_2.\end{aligned}\right\} \tag{37}$$

The local lift force $L(x,t)$ is calculated from Lavie (1970) as

$$L(x,t) = -\oint_\Sigma \{lp_1 + 2\rho\nu m\Gamma_1\}\,ds, \tag{38}$$

where l and m are the direction cosines, Σ is the contour of the cross-section, Γ_1 is the vorticity function and s is the arc length along the contour.

By substitution of the appropriate expressions for p_1 and Γ_1 in (38) one gets

$$L(x,t) = -\rho\left(\frac{\partial}{\partial t} + W_2\frac{\partial}{\partial x}\right)(C_\mathrm{I}\tilde{A}(x)\,V(x,t)) - \rho\nu c^2 V(x,t)\,C_\mathrm{II}\tilde{A}(x), \tag{39}$$

where
$$C_\mathrm{I}\tilde{A}(x) = \oint_\Sigma \phi l\,ds, \quad C_\mathrm{II} = \oint_\Sigma \psi_2 m\,ds. \tag{40}$$

In order to evaluate (40) one should find the potential ϕ and the stream function ψ_2 for each individual case.

For instance, let us take the case of an elongated body with a circular instantaneous cross-section with radius $r = a(x)$ swimming in a pipe of radius b. On the cylinder $r = a(x)$, $v = 0$ and $w = V = (\partial h/\partial t) + W_2(\partial h/\partial x)$ and on the pipe $r = b$, $v = 0$ and $w = 0$. For that case the stream functions and the potential are

$$\psi_1 = \sin\theta\{Ar^{-1} + Br\}, \quad \psi_2 = \sin\theta\{DK_1(cr) + EI_1(cr)\},$$
$$\phi = \cos\theta(Ar^{-1} + Br),$$

where $K_1(cr)$ and $I_1(cr)$ are Bessel functions with imaginary argument of the first and second type and of order one. From the boundary conditions represented above the constants A, B, D and E can be evaluated:

$$A = a^2\{K_2(ca)[b^2I_2(cb)-a^2I_2(ca)]-I_2(ca)[b^2K_2(cb)-a^2K_2(ca)]\}/\bar{\Delta},$$
$$B = K_0(cb)[b^2I_2(cb)-a^2I_2(ca)]-I_0(cb)[b^2K_2(cb)-a^2K_2(ca)]/\bar{\Delta},$$
$$D = 2[b^2I_2(cb)-a^2I_2(ca)]/c\bar{\Delta}, \quad E = 2[b^2K_2(cb)-a^2K_2(ca)]/c\bar{\Delta},$$
$$\bar{\Delta} = [K_0(cb)-K_0(ca)][b^2I_2(cb)-a^2I_2(ca)]-[I_0(cb)-I_0(ca)][b^2K_2(cb)-a^2K_2(ca)],$$

where K_0, K_2, I_0 and I_2 are Bessel functions (defined in Watson 1962, p. 79) with imaginary argument of type one and two and order zero and two.

The equivalent masses per unit length are found to be

$$C_{\mathrm{I}}\tilde{A}(X) = \oint_{\Sigma} \phi l\, ds = \left\{\frac{2A}{a^2}+2B\right\}A(X), \tag{41}$$
$$C_{\mathrm{II}}\tilde{A}(X) = \oint \psi_2 m\, ds = \left\{\frac{2DK_1(ca)}{a}+\frac{2EI_1(ca)}{a}\right\}A(X),$$

where $A(X) = \pi a^2$, the cross-sectional area.

If we let $b\to\infty$ (for the swimming of a body in an infinite fluid) it can be shown that the constants A, B, D and E satisfy

$$\underset{b\to\infty}{A} \to \{-a(x)^2K_2[ca(x)]\}/K_0[ca(x)], \quad \underset{b\to\infty}{B} \to 0,$$
$$\underset{b\to\infty}{D} \to \{2/\{cK_0[ca(x)]\}, \quad \underset{b\to\infty}{E} \to 0,$$

which are in full agreement with the results achieved by Lavie (1970).

Segel (1961) has found the stream function and the force on a vibrating cylinder surrounded by viscous flow bounded by an outer cylinder. To compare the results obtained here with those obtained by Segel, we shall take the simple case of a rigid cylinder with radius $r = a$ vibrating in a liquid bounded by an outer cylinder of radius b. In this case the lift force per unit length of the cylinder is

$$L(t) = -\rho i\Omega V(t)\pi a^2\{(2b^2/\Delta)[K_2(ca)I_2(cb)-I_2(ca)K_2(cb)]+1\},$$

where $V(t) = \mathrm{Im}\{V_1e^{i\Omega t}\}$. The term $\mathrm{Im}\{\rho i\Omega V(t)\pi a^2\}$ in the above expression is the force required to balance inertia of the inner cylinder. The above result coincides with the results obtained by Segel. Let us define the Reynolds number $Re = a^2\Omega/\nu$ such that $ca = (i\,Re)^{\frac{1}{2}}$. It can be shown that for $\Omega\to\infty$, and therefore for $Re\to\infty$, the above expression for the lift $L(t)$ on a vibrating cylinder approaches the value

$$\frac{b^2+a^2}{b^2-a^2}\rho A\Omega V_1\cos\Omega t+\frac{b^2 2\sqrt{2}}{(b^2-a^2)\sqrt{Re}}\rho A\Omega V_1\cos\Omega t+\frac{b^2 2\sqrt{2}}{(b^2-a^2)\sqrt{Re}}\rho A\Omega V_1\sin\Omega t.$$

In the limiting case when $Re = \infty$ (i.e. $\nu = 0$) $L(t)$ approaches the value for inviscid flow, for which the virtual mass was given by (17). In the above expression $A = \pi a^2$ and the terms containing $V_1\cos\Omega t$ constitute the virtual mass for the

$Re = \infty$	α	2	3	5	∞
	m	1·67	1·25	1·085	1·00
	k	0	0	0	0
$Re = 900$	α	2	3	5	∞
	m	1·795	1·366	1·183	1·094
	k	0·125	1·116	0·0988	0·094
$Re = 1$	α	2	3	5	∞
	m	1	1	1	1
	k	+42·6	13·3	5·8	0

TABLE 1

cylinder vibrating within the boundaries of the outer cylinder with radius b. The term containing $V_1 \sin \Omega t$ is actually the damping force.

When $Re \to 0$ the expression for $L(t)$ can be expanded into

$$\underset{Re\to 0}{L(t)} = -\frac{4\rho A\Omega V_1 \sin\Omega t}{\ln(b/a)\,Re-(b^2-a^2)/(b^2+a^2)}+\rho A\Omega V_1\cos\Omega t.$$

The $V_1 \cos \Omega t$ terms can be considered as the virtual equivalent mass known in inviscid theory. The $V_1 \sin \Omega t$ terms, terms which are in phase with the velocity, are actually the viscous damping force. Let us define by m the normalized virtual mass given by

$$m = \begin{cases} \dfrac{\alpha^2+1}{\alpha^2-1}+\dfrac{\alpha^2 2\sqrt{2}}{(\alpha^2-1)\sqrt{Re}} & (Re \gg 1), \\ 1 & (Re \ll 1), \end{cases}$$

where $\alpha = b/a$, and let us also define by k the normalized damping factor given by

$$k = \begin{cases} \dfrac{\alpha^2 2\sqrt{2}}{(\alpha^2-1)\sqrt{Re}} & (Re \gg 1), \\ -\dfrac{4}{Re\ln(\alpha)-(\alpha^2-1)/(\alpha^2+1)} & (Re \ll 1). \end{cases}$$

To illustrate the above expressions we summarize the results in table 1.

From the above results one can see that the virtual mass m has a maximum between $Re = \infty$ and $Re = 0$. The damping factor increases all the time as $Re \to 0$ and actually tends to infinity at $Re = 0$. Both m and k tend to infinity as $\alpha \to 1$ (i.e. $b \to a$).

As was mentioned by Segel the results achieved for $Re \ll 1$ are valid as long as αRe also tends to zero. The case $Re \ll 1$ but αRe not too small is specially treated by Segel.

The thrust force created by the local lift is

$$P_1 = \int_0^l L(x,t)\frac{\partial h(x,t)}{\partial x}dx$$
$$= -\rho\frac{\partial}{\partial t}\int_0^l VC_{\mathrm{I}}\tilde{A}\frac{\partial h}{\partial x}dx-\rho\left[W_2VC_{\mathrm{I}}\tilde{A}\frac{\partial h}{\partial x}\right]_0^l+\tfrac{1}{2}\rho[C_{\mathrm{I}}\tilde{A}V^2]_0^l-\rho\int_0^l C_{\mathrm{II}}\tilde{A}V\frac{\partial h}{\partial x}dx; \quad (42)$$

obviously the total thrust is $\qquad P = P_0+P_1. \qquad (43)$

4. Discussion of results

Comparing the results obtained when the viscosity is considered with the results obtained from potential flow (see the expressions for the local lift $L(x,t)$ and the thrust P in (21), (23), (39) and (43) and table 1) we see that there are two main differences.

(*a*) The virtual mass $\rho\tilde{A}(x)$ per unit length from potential flow (equation (16)) is replaced by an equivalent mass per unit length defined by $\rho C_{\mathrm{I}}\tilde{A}(x)$ (equation (41)) which includes effects of the viscosity and the frequency Ω of the lateral velocity or the Reynolds number $Re = a^2\Omega/\nu$. When the viscosity is equal to zero then $\rho C_{\mathrm{I}}\tilde{A}(x)$ becomes the usual virtual mass per unit length ($Re \rightarrow \infty$).

(*b*) There are additional damping terms dependent on the viscosity. These terms on one hand reduce the thrust force P and on the other hand increase the total power, causing a poorer efficiency. When the viscosity is zero, these terms become, of course, zero too.

It should be noted that the analysis represented in this article is valid as long as the radius of the pipe is considerably bigger than the cross-sectional dimensions of the body. For smaller radii one cannot accept the assumptions about the velocity distribution between the surface of the body and the inner wall of the pipe. It may be expected that the boundary layers occupy the whole space between them and no potential flow can be expected. However, it is expected that for radii at least twice or three times the cross-sectional dimension of the body the above results can be accepted.

In the case of the Pod this theory cannot be applied for swimming in the blood vessels in the brain. For other blood vessels in the human body the diameter of the vessels may be several times bigger than the diameter of the Pod.

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